

# Casimir energy density in closed hyperbolic universes\*

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## **Abstract**

The original Casimir effect results from the difference in the vacuum energies of the electromagnetic field, between that in a region of space with boundary conditions and that in the same region without boundary conditions. In this paper we develop the theory of a similar situation, involving a scalar field in spacetimes with negative spatial curvature.

## **1 INTRODUCTION**

In a previous work [1] the Casimir energy density was obtained for a Robertson-Walker (RW) cosmological model with constant, negative spatial curvature.

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Its spatial section was Weeks manifold, which is the hyperbolic 3-manifold with the smallest volume (normalized to  $K = -1$  curvature) in the SNAPPEA census [2].

Here we further develop and clarify the theoretical formalism of that paper.

Our sign conventions for general relativity are those of Birrell and Davies [3]: metric signature  $(+---)$ , Riemann tensor  $R^\alpha_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha_{\beta\gamma} - \dots$ , Ricci tensor  $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ .

## 2 THE ORIGINAL CASIMIR EFFECT

The original effect was calculated by Casimir [4]. Briefly, one sets two metallic, uncharged parallel plates, separated by a small distance  $a$ . Between them the electromagnetic field wavenumbers normal to the plates are constrained by the boundaries. So there is a difference  $\delta E$  between the vacuum energy for this configuration and the vacuum energy for unbounded space. If  $A$  is the area of each plate, one has (see, for example, [5], [6], [7])

$$\frac{\delta E}{A} \doteq \frac{\hbar c}{2} \int \int \frac{dk_x dk_y}{(2\pi)^2} \left[ \sum_{n \in \mathbb{Z}} \sqrt{k_x^2 + k_y^2 + (\pi n/a)^2} - 2a \int \frac{dk_z}{2\pi} \sqrt{k_x^2 + k_y^2 + k_z^2} \right],$$

where we omitted damping factors needed to avoid infinities. The results is

$$\delta E(a) = -\frac{\pi^2 \hbar c}{720 a^3} A$$

for the energy difference, and

$$F(a) = -\frac{\pi^2}{240 a^4} A$$

for the attractive force between the plates.

## 3 CASIMIR ENERGY (CE) IN COSMOLOGY WITH NONTRIVIAL TOPOLOGY

There is no boundary for a universe model with closed (i.e., compact and boundless) spatial sections. But a field in these models has periodicities,

which leads to an effect similar to the above one, that may also be called a Casimir effect.

A simple example, taken from Birrell and Davis [3], is that of a scalar field  $\phi(t, x)$  in spacetime  $\mathbf{R}^1 \times S^1$ , with one closed space direction. If  $S^1$  has length  $L$  then

$$\phi(t, x + L) = \phi(t, x) ,$$

and the vacuum energy density is

$$\rho = -\pi\hbar c/6L^2.$$

An analytical expression for the CE in a class of closed hyperbolic universes (CHUs) was obtained by Goncharov and Bytsenko [8].

Here we develop a formalism succinctly described in [1], for the numerical calculation of the CE *density* of closed hyperbolic universes.

Our notation:  $i, j, \dots = 1 - 3$ ;  $\alpha, \mu = 0 - 3$ ;  $\mathbf{x} = (x^i)$ ;  $x = (x^\mu) = (t, \mathbf{x})$ .

Sign conventions are those of [3]: metric signature  $(+ - - -)$ , Riemann tensor  $R^\alpha_{\beta\gamma\delta} = \partial_\delta \Gamma^\alpha_{\beta\gamma} - \dots$ , Ricci tensor:  $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ .

## 4 SCALAR FIELD $\phi(x)$ IN CURVED SPACE-TIME

The action for a scalar field in a curved spacetime of metric  $g_{\mu\nu}$  and mass  $m$  is

$$S = \int \mathcal{L}(x) d^4x ,$$

with

$$\mathcal{L} = \frac{1}{2} \sqrt{-g} [g^{\mu\nu} \phi_{;\mu} \phi_{;\nu} - (m^2 + \xi R) \phi^2] ,$$

where  $R$  is scalar curvature of spacetime,  $g = \det(g_{\mu\nu})$ , and  $\xi$  is a constant.

With  $\xi = 1/6$  (“conformal” value) we get the equation for  $\phi(x)$ :

$$\frac{\delta S}{\delta \phi} = 0 \Rightarrow (\square + m^2 + \frac{1}{6}R)\phi = 0,$$

where  $\square$  is the generalized d’Alembertian:

$$\square \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = (-g)^{-1/2} \partial_\mu [(-g)^{1/2} g^{\mu\nu} \partial_\nu \phi] .$$

The energy-momentum tensor is (cf. [3])

$$\begin{aligned} T_{\mu\nu} &= 2(-g)^{-1/2} \delta S / \delta g^{\mu\nu} \\ &= \frac{2}{3} \phi_{;\mu} \phi_{;\nu} + \frac{1}{6} g_{\mu\nu} \phi_{;\sigma} \phi^{;\sigma} - \frac{1}{3} \phi_{;\mu\nu} + \frac{1}{12} g_{\mu\nu} \phi \square \phi \\ &\quad - \frac{1}{6} R_{\mu\nu} \phi^2 + \frac{1}{24} g_{\mu\nu} R \phi^2 + \frac{1}{4} g_{\mu\nu} m^2 \phi^2 . \end{aligned}$$

## 5 COORDINATES IN $H^3$

The hyperbolic (or Bólyai-Lobachevsky) space  $H^3$  is isometric to the hyper-surface

$$(x^4)^2 - \mathbf{x}^2 = 1, \quad x^4 \geq 1 ,$$

imbedded in an abstract Minkowski space  $(\mathbf{R}^4, \text{diag}(1, 1, 1, -1))$ .

This upper branch of a hyperboloid is similar to the mass shell of particle physics,

$$E^2 - \mathbf{p}^2 = m^2, \quad E \geq m .$$

Hence a point in  $H^3$  may be represented by the Minkowski coordinates  $x^b$ ,  $b = 1 - 4$ , subject to constraints (1), and rigid motions in  $H^3$  are proper, orthochronous Lorentz transformations.

We relate the spherical coordinates  $(\chi, \theta, \varphi)$  to the displaced Minkowski ones  $x^b - x'^b$ ,  $b = 1 - 4$ :

$$\begin{aligned} x^1 - x'^1 &= \sinh \chi \sin \theta \cos \varphi , \\ x^2 - x'^2 &= \sinh \chi \sin \theta \sin \varphi , \\ x^3 - x'^3 &= \sinh \chi \cos \theta , \\ x^4 - x'^4 &= \cosh \chi . \end{aligned}$$

Note that  $\chi(\mathbf{x}, \mathbf{x}') = \sinh^{-1} |\mathbf{x} - \mathbf{x}'|$  .

## 6 STATIC MODELS OF NEGATIVE SPATIAL CURVATURE

The Robertson-Walker metric for spatial curvature  $K = -1/a^2$  is

$$\begin{aligned} ds^2 &= dt^2 - a^2(d\chi^2 + \sinh^2 \chi d\Omega^2) \\ &= dt^2 - a^2 \left( \delta_{ij} - \frac{x^i x^j}{1 + \mathbf{x}^2} \right) dx^i dx^j , \end{aligned}$$

where in general  $a = a(t)$ .

Einstein's equations give

$$\begin{aligned} \left( \frac{\dot{a}}{a} \right)^2 &= \frac{1}{a^2} + \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} , \\ \frac{3\ddot{a}}{a} &= -4\pi G(\rho + 3P) + \Lambda . \end{aligned}$$

Assuming  $\dot{a} = \ddot{a} = 0$  and  $P = \rho/3$  we get  $a^2 = -3/2\Lambda$ , hence  $\Lambda < 0$ , and

$$a = \sqrt{3/2|\Lambda|} ,$$

$$\rho = \Lambda/8\pi G < 0 .$$

We will comment below on this negative energy density.

These models are stable (!) under curvature fluctuations:

$$a \rightarrow a + \varepsilon(t) \implies \ddot{\varepsilon} + |\Lambda|\varepsilon = 0 .$$

## 7 CLOSED HYPERBOLIC 3-MANIFOLDS (CHMs)

A CHM is obtained by a pairwise identification of the  $n$  faces of a *fundamental polyhedron* ( $FP$ ), or *Dirichlet domain*, in hyperbolic space. It is isometric to the quotient space  $H^3/\Gamma$ , where  $\Gamma$  is a discrete group of isometries of  $H^3$ , defined by generators and relations, which acts on  $H^3$  so as to produce the *tessellation*

$$H^3 = \bigcup_{\gamma \in \Gamma} \gamma(FP) .$$

Each cell  $\gamma(FP)$  is a copy of  $FP$ , hence we have periodicity of functions on a CHM, and the possibility of a cosmological Casimir effect.

*Face-pairing* generators  $\gamma_k$ ,  $k = 1 - n$ , satisfy

$$FP \cap \gamma_k(FP) = \text{face } k \text{ of } FP .$$

With these generators the relations also have a clear geometrical meaning: they correspond to the *cycles* of cells around the edges of  $FP$ .

The software SNAPPEA [2] includes a “census” of about 11,000 orientable CHMs, with normalized volumes from 0.94270736 to 6.45352885. For each of these the  $FP$  centered on a special basepoint  $O$  is given, as well as the face-pairing generators in both the  $SL(2, C)$  and the  $SO(1, 3)$  representations.

An algorithm [12] to find a set of cells  $\gamma(FP)$  that completely cover a ball of radius  $r$  reduces this problem to one of finding all motions  $\gamma \in \Gamma$ , such that

$$\text{distance}[O, \gamma(O)] < r + (\text{radius of } FP\text{'s circumscribing sphere}) .$$

For a study of CHMs from a cosmological viewpoint, see for example [9] and references therein. For numerical data on a couple of them, see [10], [11].

## 8 CLOSED HYPERBOLIC UNIVERSES

We are considering *static* CHUs. As obtained in Sec. 6, the metric is

$$ds^2 = dt^2 - \frac{3}{2|\Lambda|} \left( \delta_{ij} - \frac{x^i x^j}{1 + \mathbf{x}^2} \right) dx^i dx^j .$$

The spacetimes have nontrivial topology:

$$M^4 = \mathbf{R}^1 \times \Sigma ,$$

where  $\mathbf{R}^1$  is the time axis and  $\Sigma = H^3/\Gamma$  is a CHM.

As found above, these models have negative energy density,  $\rho = \Lambda/8\pi G$ , which has no obvious physical meaning, and violates the energy condition  $T_{\mu\nu}u^\mu u^\nu \geq 0$ . But we are dealing with the very early universe, where one feels freer to speculate. And a recent paper by Olum [13] casts doubt on the universality of this condition.

Our original motivation was the possibility of preinflationary homogenization through chaotic mixing, leading to  $\Omega_0 < 1$  inflation (cf. Cornish et al. [14]).

Another guess is that these models might have a place in the path integrals for quantum cosmology.

## 9 THE ENERGY-MOMENTUM OPERATOR

If we use the equation for  $T_{\mu\nu}$  in Sec. 4 to calculate  $\langle 0|T_{\mu\nu}|0 \rangle$  we get terms like

$$\langle 0|\phi(x)\phi(x)|0 \rangle ,$$

which lead to infinities.

To avoid this one replaces  $x$  by  $x'$  in the first factor, then in the second factor, and average the result. Thus the above expectation value becomes one-half Hadamard's function  $G^{(1)}$  :

$$G^{(1)}(x, x') = \langle 0|[\phi(x), \phi(x')]_+|0 \rangle ,$$

and we obtain (cf. Christensen [15], with our signs)

$$\langle 0|T_{\mu\nu}(x, x')|0 \rangle = \hat{T}_{\mu\nu}(x, x')G^{(1)}(x, x') ,$$

with the operator

$$\begin{aligned} \hat{T}_{\mu\nu}(x, x') = & \frac{1}{6}(\nabla_\mu \nabla_{\nu'} + \nabla_{\mu'} \nabla_\nu) + \frac{1}{12}g_{\mu\nu}(x)\nabla_\rho \nabla^{\rho'} \\ & - \frac{1}{12}(\nabla_\mu \nabla_\nu + \nabla_{\mu'} \nabla_{\nu'}) + \frac{1}{48}g_{\mu\nu}(x)(\nabla_\rho \nabla^\rho + \nabla_{\rho'} \nabla^{\rho'}) \\ & - \frac{1}{12} \left[ R_{\mu\nu}(x) - \frac{1}{4}g_{\mu\nu}(x)R(x) \right] + \frac{1}{8}m^2 g_{\mu\nu}(x) , \end{aligned}$$

where  $\nabla_\alpha$  and  $\nabla_{\alpha'}$  are covariant derivatives with respect to  $x^\alpha$  and  $x'^\alpha$ , respectively.

Eventually one takes the limit  $x \rightarrow x'$  to get the CE density. But first we have to investigate  $G^{(1)}(x, x')$ .

## 10 FEYNMAN'S PROPAGATOR

$G^{(1)}(x, x')$  will be obtained from Feynman's propagator for a scalar field  $G_F(x, x')$ .

In an  $\mathbf{R}^1 \times H^3$  universe,  $G_F$  gets an extra factor  $(\chi/\sinh \chi)$ , where  $\chi = \sinh^{-1} |\mathbf{x} - \mathbf{x}'|$ , with respect to its flat spacetime counterpart; and the squared interval  $(x - x')^2$  in the latter becomes  $2\sigma = (t - t')^2 - a^2 \chi^2$ , which is the squared geodesic distance between  $x$  and  $x'$ . The derivation of the following expression (with opposite sign because of a different metric signature) is outlined in [1]:

$$G_F(x, x') = \frac{m^2}{8\pi} \frac{\chi}{\sinh \chi} \frac{H_1^{(2)}(m\sqrt{2\sigma})}{m\sqrt{2\sigma}},$$

where  $H_1^{(2)}$  is Hankel's function of second kind and degree one.

For our spacetime  $\mathbf{R}^1 \times H^3/\Gamma$ , point  $\mathbf{x}$  may be reached by the projections of all geodesics that link  $\mathbf{x}'$  to  $\gamma\mathbf{x}$  in the covering space  $H^3$  [in Minkowski coordinates  $(\gamma\mathbf{x})^i = \sum_{b=1}^4 \gamma^i_b x^b$ ,  $i = 1 - 3$ ].

Therefore our propagator is

$$G_F(x, x') = \frac{m^2}{8\pi} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\sinh \chi(\gamma)} \frac{H_1^{(2)}(m\sqrt{2\sigma(\gamma)})}{m\sqrt{2\sigma(\gamma)}},$$

with  $\chi(\gamma) = \sinh^{-1} |\gamma\mathbf{x} - \mathbf{x}'|$  and  $2\sigma(\gamma) = (t - t')^2 - a^2 \chi^2(\gamma)$ .

## 11 HADAMARD'S FUNCTION

Hadamard's function is related to  $G_F$  and the principal value Green's function  $\bar{G}$  by

$$G_F(x, x') = -\bar{G}(x, x') - \frac{i}{2} G^{(1)}(x, x').$$

In our problem, both  $\bar{G}$  and  $G^{(1)}$  are real, so that

$$G^{(1)}(x, x') = -2 \operatorname{Im} G_F(x, x').$$



We need  $G^{(1)}(x, x')$  for  $x'$  near  $x$ , hence when  $2\sigma(\gamma)$  is near  $-a^2\chi^2 \leq 0$ . So we write the argument of  $H_1^{(2)}$  as  $iu_\gamma$ , with  $u_\gamma = m\sqrt{2|\sigma(\gamma)|}$ . From the properties of Bessel functions,

$$-2 \operatorname{Im} \left[ (iu_\gamma)^{-1} H_1^{(2)}(iu_\gamma) \right] = (4/\pi) u_\gamma^{-1} K_1(u_\gamma) ,$$

where  $K_1$  is a modified Bessel function of degree one.

Hadamard's function for a universe  $\mathbf{R}^1 \times H^3/\Gamma$  is then

$$G_\Gamma^{(1)}(x, x') = \frac{m^2}{2\pi^2} \sum_{\gamma \in \Gamma} \frac{\chi(\gamma)}{\sinh \chi(\gamma)} \frac{K_1(u_\gamma)}{u_\gamma} ,$$

The  $\gamma = 1$  term in this sum corresponds to the infinite  $\mathbf{R}^1 \times H^3$  universe. Similarly to what was done for the two-plate Casimir effect in Sec. 2, we subtract it out to get a finite energy density. Therefore the expression in Sec. 9 for  $\langle 0|T_{\mu\nu}(x, x')|0 \rangle$  leads to

$$\langle 0|T_{\mu\nu}(x, x')|0 \rangle_C = \hat{T}(x, x') G_C(x, x') ,$$

where  $G_C = G_\Gamma^{(1)} - G_{\{1\}}^{(1)}$ .

## 12 THE CASIMIR ENERGY DENSITY

Finally, the CE density is given by

$$\langle 0|T_{00}(\mathbf{x})|0 \rangle_C = \lim_{x' \rightarrow x} \hat{T}_{00}(x, x') G_C(x, x') ,$$

where

$$G_C(x, x') = \frac{m^2}{2\pi^2} \sum_{\gamma \in \Gamma - \{1\}} \frac{\sinh^{-1} |\gamma \mathbf{x} - \mathbf{x}'|}{|\gamma \mathbf{x} - \mathbf{x}'|} \frac{K_1(m\sqrt{-2\sigma(\gamma)})}{m\sqrt{-2\sigma(\gamma)}} ,$$

with  $-2\sigma(\gamma) = a^2(\sinh^{-1} |\gamma \mathbf{x} - \mathbf{x}'|)^2 - (t - t')^2$ .

Looking at the expressions for  $\hat{T}_{00}(x, x')$  and  $G_C(x, x')$ , one sees they are pretty complicated.

Now enters the power of computers!

Calculations were performed by one of us (DM), for a grid of points  $(\theta, \varphi)$  on a sphere of radius  $r$  inside the  $FP$ , for a number of static CHUs. In [1] the parameters are, in Planckian units,  $m = 0.5$ ,  $a = 10$ , and  $r = 0.6$ , and the summation for  $G_C(x, x')$  contains a few thousand terms; the obtained density values oscillate around  $-2.65 \times 10^{-6}$ . New results will be published elsewhere.

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